

# Temperature dependent Chern number of topological Boson-Fermion Pairing model

Tieyan Si

*Academy of Fundamental and Interdisciplinary Sciences,  
Harbin Institute of Technology, Harbin, 150080, China*

(Dated: May 23, 2014)

We construct a quadratic Hamiltonian model of boson-fermion pairing to study its nontrivial topological Chern numbers. This model has essential relationship with topological insulator model. The valence band and conduction band are connected by linear dispersion due to the imbalanced kinetic energy between fermion and boson. This imbalance determines the biased velocity of quasi-excitation. Density of states for quasi-excitation diverges at the maximal imbalanced boson-fermion kinetic energy. The minimal energy gap for exciting up a quasiparticle closes at a critical temperature which decays exponentially with respect to the ratio of fermion's kinetic energy to boson's kinetic energy. While the topological Chern number remains a finite constant value at high temperature. Boson-fermion pairing maybe is observable in boson-fermion mixtures in optical lattice or in quark-gluon plasma.

PACS numbers: 03.75.Hh, 05.30.Fk, 74.20.De, 74.20.Rp, 03.65.Vf

## I. INTRODUCTION

Boson-Fermion mixture has many interesting physics property in ultracold atoms and quark gluon plasma. It was theoretically suggested that two of the three quarks inside a proton may form color Cooper pair. The composite boson of color Cooper pair and the rest quark maybe can exist as boson-fermion mixture [1]. Storozhenko find a T-matrix method to investigate the transition from a Fermi gas of three quarks to boson-fermion mixture of quarks [2]. The superfluid of  $^3\text{He}$ - $^4\text{He}$  mixture is another familiar boson-fermion mixture. In recent years, boson-fermion mixture of ultracold atom is already implementable in laboratory. The ultracold atom in optical lattice provides a promising way to study different phases of boson-fermion mixture of 40K and 41K atoms [3]. The bosons in the spinless boson-fermion mixture in optical lattice behaves like supersolid [4]. Boson-fermion mixture can also be implemented outside scope of the optical lattice techniques [5]. A recent numerical study of Bose-Fermi Hubbard model confirms a weak-coupling pairing mode in a mixture of boson and spin-polarized fermions [6]. The non-relativistic boson and fermion mixture bears a sharp fermionic collective mode similar to the Goldstino mode due to supersymmetry [7]. At low temperatures, the weakly interacted boson and fermion may comes into phase separation [8]. The boson cloud in boson-fermion mixture may collapse due to variation of the fermion-mediated interaction between boson and fermion [9]. The fermion cloud is also possibly collapse for an attractive fermion-boson interactions [9].

In topological insulator material, such as  $\text{HgTe/CdTe}$ ,  $\text{Bi}_2\text{Te}_3$  and  $\text{Sb}_2\text{Te}_3$ , interesting topological gapless edge states coexist with a gapped bulk state [10][11]. These topological states of topological insulator model is most likely being constructed in ultracold atoms in optical lattice [12], so does the p-wave fermion pairing superconductor model. Thus we conjectured the nontrivial topological physics may also exist in a boson-fermion

pairing model. Boson-fermion pairing was only mentioned in quark gluon plasma system before. While we try to construct a quantum many body model similar to Bardeen-Cooper-Schrieffer(BCS) pairing model. Since ultracold atom system has great potential to implement many quantum models. A boson-fermion pairing model also has a promising future. We focus on the quadratic Hamiltonian of boson-fermion pairing to study its critical temperature for gap closing point as well as its non-trivial topological Chern number at high temperature.

## II. THE BOSON-FERMION PAIRING MODEL

We construct the quadratic Hamiltonian of boson-fermion pairing model along the same line of the mean-field theory of Bardeen-Cooper-Schrieffer(BCS) pairing model. This Hamiltonian describes a composite pair of a spinless boson and a spinless fermion. The pair of boson and fermion is generated and annihilated simultaneously. The boson has an opposite momentum vector as fermion. One possible example of this kind of pairing is the three quarks confined in elementary particles like proton, neutron, et al. Single quark is fermion while a pair of bounded two quarks is a boson. Three quarks can be viewed as a composite pair of boson and fermion. For the simplest case, the quadratic Hamiltonian of boson-fermion pairing reads

$$H = \sum_{k_i} [\epsilon_{k,b} b_k^\dagger b_k + \epsilon_{k,f} f_{-k}^\dagger f_{-k}] + \Delta_{bf}^* \Delta_{bf} - \sum_k (\Delta_{bf}^* b_k f_{-k} + \Delta_{bf} b_k^\dagger f_{-k}^\dagger). \quad (1)$$

The minimal energy for exciting up a boson-fermion pair is  $|\Delta_{bf}|$ . Usually the order parameter  $\Delta_{bf}$  is called energy gap. In the framework of Ginzburg-Landau theory, the gap function  $\Delta_{bf}$  is the wave function of boson-fermion pairs. To meet the physical reality, we require that  $\Delta_{bf}$

is an ordinary real function instead of Grassmann number here. The particle number of the composite particle of boson-fermions pair is denoted by  $\Delta_{bf}^* \Delta_{bf} = |\Delta_{bf}|^2 = N_{bf}$ . This particle number must be a real number. The total particle number of this boson-fermion pairing model is not conserved. Similar to the supersymmetric model of Boson-Fermion mixture [7], we can also introduce a supersymmetry operator,  $Q = \sum_k b_k^\dagger f_k$ ,  $Q^\dagger = \sum_k b_k f_k^\dagger$ . The two supersymmetric operator obey the commutator,  $\{Q, Q^\dagger\} = \sum_k (b_k^\dagger b_k + f_k^\dagger f_k) = N_{total}$ . However this supersymmetry operator does not commute with the boson-fermion pairing Hamiltonian,  $[Q, H] \neq 0$ . This model itself does not keep supersymmetry. However a closer observation on the Hamiltonian still illustrates a hidden symmetry. If the boson and fermion have the same dispersion,  $\epsilon_{k,b} = \epsilon_{k,f}$ , and in the meantime we perform a time reversal transformation on momentum vector,  $k \rightleftharpoons -k$ , then the Hamiltonian would keep the same formulation as before. This symmetry maybe can be called time reversal symmetry jointed with supersymmetry.

The eigenenergy of quasi-excitations was calculated for the most general case without distinguishing the relativistic particle from nonrelativistic particle. The kinetic energy of the boson and fermion here has a general formulation,  $\epsilon_{k,f} = \alpha_f \epsilon_0(k)$ ,  $\epsilon_{k,b} = \alpha_b \epsilon_0(k)$ .  $\epsilon_0(k)$  is a function of momentum vector which has different formulations for different models. The eigen-energy of quasi-particle is obtained by Green-function method,

$$\begin{aligned} E_1 &= [\alpha_- \epsilon_0 - \sqrt{[\alpha_+ \epsilon_0]^2 - \Delta^2}] , \\ E_2 &= [\alpha_- \epsilon_0 + \sqrt{[\alpha_+ \epsilon_0]^2 - \Delta^2}] . \end{aligned} \quad (2)$$

Here the coefficient  $\alpha_\pm$  are  $\alpha_- = \alpha_b - \alpha_f$  and  $\alpha_+ = \alpha_f + \alpha_b$ . The gap function is  $\Delta^2 = (2n_{f,-k} - 1)(\Delta_{bf}^* \Delta_{bf})$ .  $\Delta_{bf}^* \Delta_{bf} = |\Delta_{bf}|^2 = N_{bf}$  is real number which indicates the density distribution of boson-fermion pairs. Notice here the energy gap for exciting up a pair of boson-fermion depends on the occupation of fermion. This is a key difference from that case of BCS model. The gap is negative when the occupation of fermions becomes zero,  $n_{f,k} = 0$ . The gap is positive if  $n_{f,k} = 1$ . The gap is closed at half-filling state,  $n_{f,k} = 1/2$ .

The spectrum of the boson-fermion pairing Hamiltonian obtained by Green function method (Fig. 1) has two separated energy bands. The red curve denotes the real part of the eigenenergy. The dash line labels the imaginary part of eigenenergy. The imaginary part of the spectrum forms a closed loop. The fermi energy is supposed at zero energy point. The energy band below zero is valence band. The energy band above zero is conduction band. A linear dispersion bridges the gap between conduction band and valence band. The linear dispersion has two fold degeneracy. The linear dispersion vanishes when the boson has the same dispersion as fermion,  $\epsilon_{k,b} = \epsilon_{k,f}$ , then it becomes a fully gapped spectrum. Thus it is the imbalanced kinetic energy between

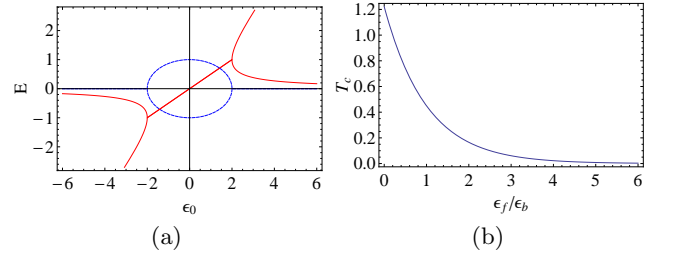


FIG. 1: (a) The spectrum of the eigenexcitations of composite particle. The dash line indicates the imaginary part of Hamiltonian. Here the occupation of fermions is above 1/2. (b) The dependence of critical temperature for gap closing point on the ratio of fermionic kinetic energy to bosonic kinetic energy.

boson and fermion that generates the linear dispersion.

The phase transition from a gapped phase to gapless phase of this boson-fermion pairing model is different from that of BCS model. The gap of boson-fermion pair is the product of the occupation of fermion and conventional gap similar to that of Cooper pairs. The boson-fermion pairing gap at finite temperature is calculated by Green function theory,

$$\Delta^* = \sum_k V D_1 \left[ \frac{1}{e^{\beta E_1} + 1} - \frac{1}{e^{\beta E_2} + 1} \right], \quad (3)$$

where the coefficient  $D_1$  is

$$D_1 = \frac{(2n_{f,-k} - 1)V\Delta^*}{\sqrt{(\epsilon_{k,b} + \epsilon_{k,f})^2 - 4(2n_{f,-k} - 1)N_{bf}}}. \quad (4)$$

The gap is closed at the critical temperature for which we set  $\Delta = 0$ , then the self-consistent equation above reduces to

$$1 = \sum_k V \frac{(2n_{f,-k} - 1)}{(\epsilon_{k,b} + \epsilon_{k,f})} \left[ \frac{1}{e^{\beta \epsilon_{k,b}} + 1} - \frac{1}{e^{-\beta \epsilon_{k,f}} + 1} \right], \quad (5)$$

here  $n_{f,-k}$  is the Fermi-Dirac distribution,  $n_{f,-k} = (e^{\beta \epsilon_f} + 1)^{-1}$ . For non-relativistic particle, the ratio of fermion's kinetic energy to the boson's kinetic energy is inversely proportional to the ratio of fermion's mass and boson's mass,  $\epsilon_{k,f}/\epsilon_{k,b} = m_b/m_f$ . The critical equation can be transformed into an integral equation with two parts. We made a cutoff at the Debye frequency of fermi metal in the first part of the integral equation, i.e.,  $\hbar \epsilon_D \gg k_B T$ , then  $(2n_{f,-k}(\beta \epsilon_D) - 1) = -\tanh[\frac{\beta \epsilon_D}{2}]$ . The value of the occupation function of fermions are  $\tanh[\infty] = 1, \tanh[0] = 0$ . The second part of the integral equation contributes a constant. This constant number is 0.228788 for the special case of  $(\epsilon_{k,f}/\epsilon_{k,b})=2$  and  $\beta = 3$ . For this special case, the critical temperature derived from the integral equation reads,  $[(1 + (\epsilon_{k,f}/\epsilon_{k,b})) + 0.228788] = V \ln[\beta \epsilon_D]$ . The critical temperature obeys similar relation as BCS model,

$$T_c = \frac{\epsilon_D}{k_B} e^{-\frac{1 + (\epsilon_{k,f}/\epsilon_{k,b}) + 0.228788}{V}}. \quad (6)$$

The constant number 0.228788 fluctuates from 0.18 to 0.55 when the parameter pairs  $[(\epsilon_{k,f}/\epsilon_{k,b}), \beta]$  varies in a wide range. Thus the critical temperature does not become divergent. Fig. 1 (b) shows the exponential decay of critical temperature with an increasing ratio of fermion kinetic energy to boson kinetic energy. If the boson and fermion are nonrelativistic particle, then  $\epsilon_{k,f}/\epsilon_{k,b} = m_b/m_f$ . The critical temperature grows higher for a heavier boson or lighter fermion. Further more, there is a special temperature that determined by the occupation of fermion. The gap is always closed in the half-filling state of fermion. That critical temperature can be directly read out from the Fermi-Dirac distribution.

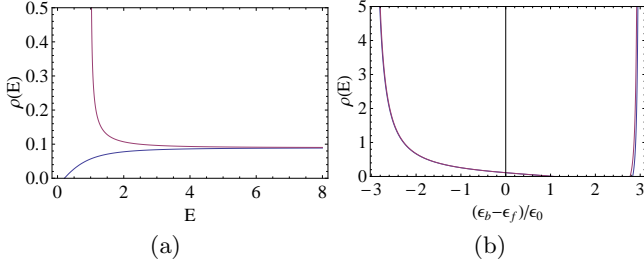


FIG. 2: (a) Density of states for quasi-excitation.  $\alpha_- = 0.2$ ,  $\alpha_+ = 3$ . The red curve corresponds to  $n_f = 1$ , the blue curve corresponds to  $n_f = 0$ . (b) Density of states for quasi-excitation.  $\alpha_+ = 3$ ,  $E = 8$ ,  $\Delta = 1$ . The red curve corresponds to  $n_f = 1$ , the blue curve corresponds to  $n_f = 0$ .

The density of state for BCS model turns out to be one extreme case of this boson-fermion pairing model. With the exact dispersion of quasi-excitation spectrum Eq. (2), we calculate their density of state,  $\rho(E) = (dE/d\epsilon)^{-1}$ ,

$$\rho(E) = \frac{\alpha_-}{\alpha_-^2 - \alpha_+^2} + \frac{(\alpha_+^2 E)/(\alpha_-^2 - \alpha_+^2)}{[(\alpha_+^2 - \alpha_-^2)\Delta^2 + \alpha_+^2 E^2]^{1/2}}. \quad (7)$$

When the occupation of fermion is  $n_f = 1$ , and ( $\alpha_- < 0$ ,  $\alpha_+ > 0$ ,  $\alpha_+^2 - \alpha_-^2 > 0$ ), the density of state is divergent as a square root singularity. This is exactly the case of BCS model. In the other case, if the occupation of fermion is zero,  $n_f = 0$ , the density diverges slowly in the opposite orientation of square root singularity. Fig. 2 (b) shows behavior of the density of state for different ratio of boson kinetic energy mass and fermion kinetic energy. The density of states diverges at two extremal cases: either the kinetic energy of boson becomes zero or the kinetic energy of fermion vanishes. For the average case with both nonzero bosonic kinetic energy and fermionic energy, there is still a long tail of nonzero density of states (Fig. 2 (b)). The dependence of density of state on the imbalance boson-fermion kinetic energy is not influenced by the occupation of fermion. The curve of density of states for  $n_f = 0$  and  $n_f = 1$  almost overlap each other (Fig. 2 (b)).

Topological physics has an universal existence in quantum many body system. The BCS fermion pairing model

can be diagonalized by the composite particle of electrons and holes. The operator of the composite-particle is a sum of annihilation operator and a generation operator due to Bogoliubov transformation. Here we can also define a similar composite particle,  $\alpha^\dagger = u_k^* b_k^\dagger - v_k^* f_{-k}$ ,  $\alpha = u_k b_k - v_k f_{-k}^\dagger$ . This composite operator is the combination of a fermion and a boson. However it does not obey the anti-commutator of typical fermions. Instead it obeys a complicate commutator,

$$[\alpha, \alpha^\dagger] = u_k u_k^* + v_k v_k^* (2n_{f,-k} - 1). \quad (8)$$

This commutator relation depends on the occupation of fermion. If  $n_{f,-k} = 0$ , the composite particle  $\alpha$  is a boson, it requires  $u_k u_k^* - v_k v_k^* = 1$ . If  $n_{f,-k} = 1$ , it behaves like a conventional fermion,  $u_k u_k^* + v_k v_k^* = 1$ . If  $0 < n_{f,-k} < 1$ , the composite particle is neither fermions nor Boson, it is some kind of exotic composite particle. To the first order approximation, the eigen-energy of this composite particle is  $E_\pm$  in Eq. (2).

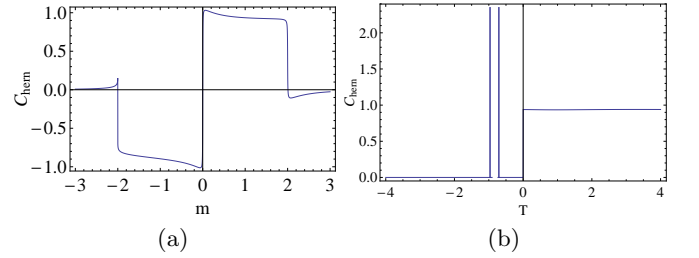


FIG. 3: (a) The Chern number for different  $m$ . (b) The Chern number for  $m=1$ . The Chern number is a constant less than one for different temperature  $T$  above zero temperature.

This boson-fermions pairing model has deep connections with topological superconductor model and topological insulator model. We define a spinor wave vector,  $\psi^\dagger = [b_k^\dagger, f_{-k}]^T$ , to reformulate the Hamiltonian,

$$H_{(bf)} = \psi^\dagger H(k) \psi + \epsilon_{k,f} + \Delta_{bf}^* \Delta_{bf}, \quad (9)$$

Then the Hamiltonian can be expressed into a matrix,

$$H(k) = \begin{bmatrix} \epsilon_{k,b} & 0 \\ 0 & -\epsilon_{k,f} \end{bmatrix} - \Delta_{bf}^1 \sigma_x + \Delta_{bf}^2 \sigma_y + \Delta_{bf}^* \Delta_{bf}. \quad (10)$$

We combine the difference between the kinetic energy of boson and fermion with the constant term  $\Delta_{bf}^* \Delta_{bf}$ , and incorporate them into the coefficient of 2 by 2 identity matrix. The complex gap function is decomposed as two real numbers in order to construct the spin operator of Pauli matrices,  $\Delta_{bf} = \Delta_{bf}^1 + i\Delta_{bf}^2$ ,  $\Delta_{bf}^* = \Delta_{bf}^1 - i\Delta_{bf}^2$ . Finally the Bose-fermion pairing Hamiltonian reaches a brief expression,

$$H = d_0 \mathbb{I} + \left( \frac{\epsilon_{b,f} + \epsilon_{k,f}}{2} \right) \sigma_z - \Delta_{bf}^1 \sigma_x + \Delta_{bf}^2 \sigma_y. \quad (11)$$

This Hamiltonian can be mapped into the topological insulator model [13]. The coefficient of the identity matrix reads  $d_0 = (\epsilon_{k,b} - \epsilon_{k,f})/2 + \Delta_{bf}^* \Delta_{bf}$ . However the eigenenergy of Hamiltonian Eq. (11) is  $E' =$

$d_0 \pm \sqrt{[(\epsilon_{b,f} + \epsilon_{k,f})/2]^2 + |\Delta_{bf}|^2}$ . This is not the exact eigenenergy of the quasiexcitation of the bosonfermion pairing model. In order to meet the exact eigenenergy of quasi-excitations, the vector field  $\mathbf{d} = (d_x, d_y, d_z)$  must be modified by the occupation of fermion. The density distribution of the total number of boson-fermion pairs was decomposed into two anisotropic parts,  $N_{bf} = |\Delta|^2 = N_{bf,1} + N_{bf,2}$ , i.e.,  $N_{bf,1}^{1/2} = \Delta_{bf}^1$ ,  $N_{bf,2}^{1/2} = \Delta_{bf}^2$ . The new vector field after the modification is

$$\begin{aligned} d_x &= -\sqrt{\tanh\left[\frac{\beta\epsilon_f}{2}\right]}[N_{bf,1}]^{1/2}, \quad d_z = (\epsilon_{k,b} + \epsilon_{k,f})/2, \\ d_y &= \sqrt{\tanh\left[\frac{\beta\epsilon_f}{2}\right]}[N_{bf,2}]^{1/2}. \end{aligned} \quad (12)$$

The effective Hamiltonian for the quasiexcitations now is  $H = d_0\mathbb{I} + d_z\sigma_z + d_x\sigma_x + d_y\sigma_y$ . The eigenenergy of this effective Hamiltonian corresponds to the exact eigenenergy of quasi-excitations. Here the well known Chern number also provides a topological quantity to reveal the topological physics of Boson-fermion pairing model,

$$C_{hern} = \frac{1}{4\pi} \int dk_x \int dk_y \mathbf{n} \cdot (\partial_{k_x} \mathbf{n} \times \partial_{k_y} \mathbf{n}). \quad (13)$$

Here we define the unit vector field as  $\mathbf{n} = (d_x/|d|, d_y/|d|, d_z/|d|)$ . The energy gap between the two branches of quasi-excitations reads  $\Delta E = E_2 - E_1$ ,  $\Delta E = \sqrt{(\epsilon_{k,b} + \epsilon_{k,f})^2 + 4(1 - 2n_{f,-k})N_{bf}}$ , where  $N_{bf} = \Delta_{bf}^* \Delta_{bf}$  is a real number indicating the density distribution of boson-fermion pairs.  $n_{k_f}$  is the Fermi-Dirac distribution,  $n_{k_f} = [e^{\beta\epsilon_{k,f}} + 1]^{-1}$ . In the vector Eq. (12), we have an identity of Fermi-Dirac distribution,  $(2n_{f,-k}(\beta\epsilon_f) - 1) = -\tanh[\frac{\beta\epsilon_f}{2}]$ . The energy gap is exactly the absolute value of the vector field  $\mathbf{d}$ ,  $\Delta E = |\mathbf{d}| = (d_x^2 + d_y^2 + d_z^2)^{1/2}$ . The Chern number has many interesting bifurcation phenomena according to Duan's topological current theory [14].

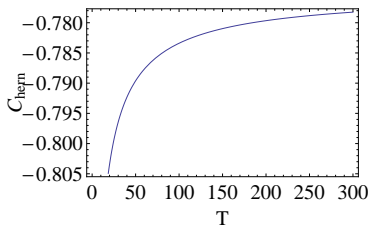


FIG. 4: The Chern number decays slowly as temperature arise for  $m=-1$ .

First we choose the same odd-even parity as the topological insulator model [13] for the gap function,  $N_{bf,1} = \sin[k_x]^2$ ,  $N_{bf,2} = \sin[k_y]^2$ ,  $d_z = \alpha_f(m + \cos[k_x] + \cos[k_y])$ . The Chern number for the vector field Eq. (12) is plotted in Fig. 3. The Chern Number is approximately one for the chemical potential of  $2 > m > 0$ . For  $0 > m > -2$ ,

the Chern number is approximately -1. The Chern number is almost zero for other cases. This is generally in consistent with the topological insulator model [13]. However, the absolute value of the Chern number is not exactly  $\pm 1$ , the absolute value of Chern number is smaller than 1. At low temperature from  $T=0$  to  $T=4$ , Chern number has a small shoulder that decays away from 1 for a growing  $m$  [Fig. 3]. For a fixed value of  $m=1$ , the Chern number is a constant smaller than 1 above zero temperature. The Chern number has two infinite peak in the zone of negative temperature, these two peaks maybe is due to some mathematical singularity instead of physical reason. We computed the Chern number for different finite temperatures. The Chern number is rather robust at high temperature. For a negative  $m = -1$ , the exact Chern number is -0.805, while it decays only from -0.805 to -0.780 in wide range of temperature from 0 to 300.

The velocity of the quasiexcitation is determined by equation,  $v = i\hbar\partial_k E$ . The linear dispersion determines the motion of the quasi-excitation. This linear dispersion is essentially rooted in the imbalanced kinetic energy between boson and fermion. If the boson has bigger kinetic energy than fermion, then the velocity is positive, other wise, the velocity is negative. If the boson and fermions are relativistic particle, then the dispersion linearly depends on momentum vector  $\epsilon_{0,b} = c\gamma k + \hat{\eta}m_b c^2$ ,  $\epsilon_{0,f} = c\gamma k + \hat{\eta}m_f c^2$ . In the tight-binding model, the dispersion reads  $\epsilon_{k,b(f)} = -\mu_{b(f)} - 2t_{b(f)}[\cos(k_x) + \cos(k_y)]$ . The relativistic system of boson-fermion system is most likely find a physical corresponding in quark-gluon plasma. The tight-binding model is also possible to be implemented by a boson-fermion mixture system in optical lattice.

### III. SUMMARY

Boson fermion mixture has hybrid physics of boson gas and fermion gas. Boson-fermion pairing states is possibly exist for three quarks in proton, neutron, and so on. We proposed the bose-fermion pairing Hamiltonian in this article. The nontrivial topological property of this bose-fermion pairing model depends on the occupation of fermions at finite temperature. The energy gap for exciting up a pair of boson and fermion becomes zero at half filling state of fermion. The critical temperature for gap closing point exponentially decays with respect to an increasing fermionic kinetic energy and an decreasing bosonic kinetic energy. If the fermion and boson are non-relativistic particle, the ratio of fermionic kinetic energy to bosonic kinetic energy is proportional to that of boson's mass to fermion's mass. Thus the critical temperature is higher for heavier boson. The imbalance between bosonic kinetic energy and fermionic kinetic energy determines the velocity of quasiexcitation. The Chern number can quantify the gapped phase and gapless phase of this boson fermion pairing model. This topological number keeps a constant value less than 1 at high temperature, thus it is rather robust at high temperature.

**Acknowledgment** This work is supported by Fundamental Research Foundation for the Central Uni-

versities and "National Natural Science Foundation of China"(Grant No. 11304062).

- 
- [1] K. Rajagopal and F. Wilczek, e-print hep-ph/0011333; M. Alford, Annu. Rev. Nucl. Part. Sci. **51**, 131 (2001).
  - [2] A. Storozhenko, P. Schuck, T. Suzuki, H. Yabu, and J. Dukelsky, Phys. Rev. A **71**, 063617 (2005).
  - [3] K. Günter, T. Stoferle, H. Mortiz, M. Kohl, and T. Esslinger, Phys. Rev. Lett. **96**, 180402 (2006).
  - [4] P. P. Orth, D. L. Bergman, and K. L. Hur, Phys. Rev. A **80**, 023624, 2009.
  - [5] A. G. Truscott et al., Science **291**, 2570 (2001).
  - [6] X. Barillier-Pertuisel, S. Pittel, L. Pollet, and P. Schuck, Phys. Rev. A **77**, 012115 (2008).
  - [7] Y. Yu and K. Yang, Phys. Rev. Lett. **100**, 090404, 2008.
  - [8] H. P. Büchler and G. Blatter, Phys. Rev. A **77**, 012115 (2008).
  - [9] S. T. Chui and V. N. Ryzhov, Phys. Rev. A **69**, 043607 (2004).
  - [10] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
  - [11] X. L. Qi and S. C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
  - [12] I. I. Satija, E. H. Zhao, Chapter 12, M. Mohan (ed.), *New Trends in Atomic and Molecular Physics*, pringer Series on Atomic, Optical, and Plasma Physics 76, 201 (2013).
  - [13] X. L. Qi, T. L. Hughes and S. C. Zhang, Phys. Rev. B **78**, 195424 (2008).
  - [14] Y. S. Duan, S. Li, G. H. Yang, Nucl. Phys. B, **514**, 705-720, (1998).